THE EVOLUTION OF FINITE-AMPLITUDE LOCALIZED VORTICES IN LINEAR SHEAR FLOWS

Michael Karp[1], Ilia Shukhman[2], Jimmy Philip[1], Jacob Cohen[1]

[1] *Faculty of Aerospace Engineering Technion - Israel Institute of Technology Haifa 32000, Israel*
[2] *Institute of Solar-Terrestrial Physics Russian Academy of Sciences, Siberian Branch, Irkutsk, 664033, P.O.Box 4026, Russia*

Recently it has been demonstrated that the generation of counter rotating vortex pair and hairpin vortices (that are frequently observed in turbulent shear flows) can be attributed to the interaction of a localized vortical disturbance and the shear of the surrounding base flow. Here we study the origin and evolution of finite-amplitude localized vortical disturbances in flows having homogeneous shear. An analytical-based method is developed that can solve the governing equations in a novel way in Fourier space with Lagrangian co-ordinates. When possible, the new results are compared with previous ones.

1 Introduction

During the last fifty years it has become clear that although wall bounded turbulent shear flows are characterized by unsteady, seemingly chaotic motion, in fact, however, the motion is not random and it has been observed to be governed by well organized coherent vortical structures. Recently, combining experimental, numerical and theoretical efforts, we were able to demonstrate [(1), (2), (3), (4) and (5)] that a simple model, which takes into account only the interaction between a localized vortical disturbance and a laminar shear base flow, is capable of reproducing the generation process and characteristics of coherent structures (counter-rotating vortex pairs (CVPs), streaks and hairpins), naturally occurring in fully developed wall bounded and free turbulent shear flows. This is schematically shown in figure 1. For the case in which the vortical disturbance is superimposed on pure shear flow our results have shown that independent of the initial disturbance geometry a small amplitude vortical disturbance eventually evolves into a pair of streamwise vortices, whereas, a sufficiently large amplitude disturbance evolves into a hairpin (or a packet of hairpins). Moreover, other characteristics such as the spanwise separation between the two elongated vortical regions (expressed in terms of wall units), the inclination angle of the hairpins and their convective velocity correspond well to those observed in turbulent bounded shear flows. Similarly when a vortex dipole is placed in a stagnation flow (or pure extensional flow), like the one existing in the braid region of a turbulent mixing layer (shown by the red arrows) a pair of CVP is developed along the principal axis.

1.1 Objective

Following (2) the objective of our work is to develop an accurate and an efficient analytical-based model which can predict the nonlinear evolution of a localized dipole vortex in base flows having an arbitrary homogeneous shear.

2 Mathematical approach

We consider the evolution of a finite amplitude localized vortical disturbance in homogeneous planar shear flow. For the general case the base flow velocity and vorticity fields (in a cartesian system, x, y and z) can be respectively written as

$$\boldsymbol{V} = \left(-\tfrac{1}{2}\,(\Omega+\sigma)\,y,\; -\tfrac{1}{2}\,(\sigma-\Omega)\,x,\; 0\right), \quad \boldsymbol{\Omega} = (0,\,0,\,\Omega),$$

where σ and Ω are two constants representing the shear and vorticity of the base flow. The base flow is respectively hyperbolic, elliptic or pure shear (Couette flow) if $\sigma > \Omega$, $\sigma < \Omega$ or $\sigma = \Omega$. The equation describing the evolution of a 3D vortex ($\boldsymbol{\omega}$) in incompressible viscous base flow is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{V} - (\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = \eta\,\Delta\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega},$$

where, η is the dynamic viscosity and the disturbance vorticity is given by $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$. The equations are Fourier transformed and converted into Lagrangian co-ordinates in Fourier space. The resulted equations are,

$$\frac{d\zeta_1(\boldsymbol{q},t)}{dt} = -\frac{1}{2}\,(\sigma-\Omega)\,\zeta_2(\boldsymbol{q},t) - \Omega\,\frac{k_1\left[\,k_1\,\zeta_2(\boldsymbol{q},t) - k_2\,\zeta_1(\boldsymbol{q},t)\right]}{k^2} - \eta\,k^2\,\zeta_1(\boldsymbol{q},t) + N_1(\boldsymbol{q},t),$$

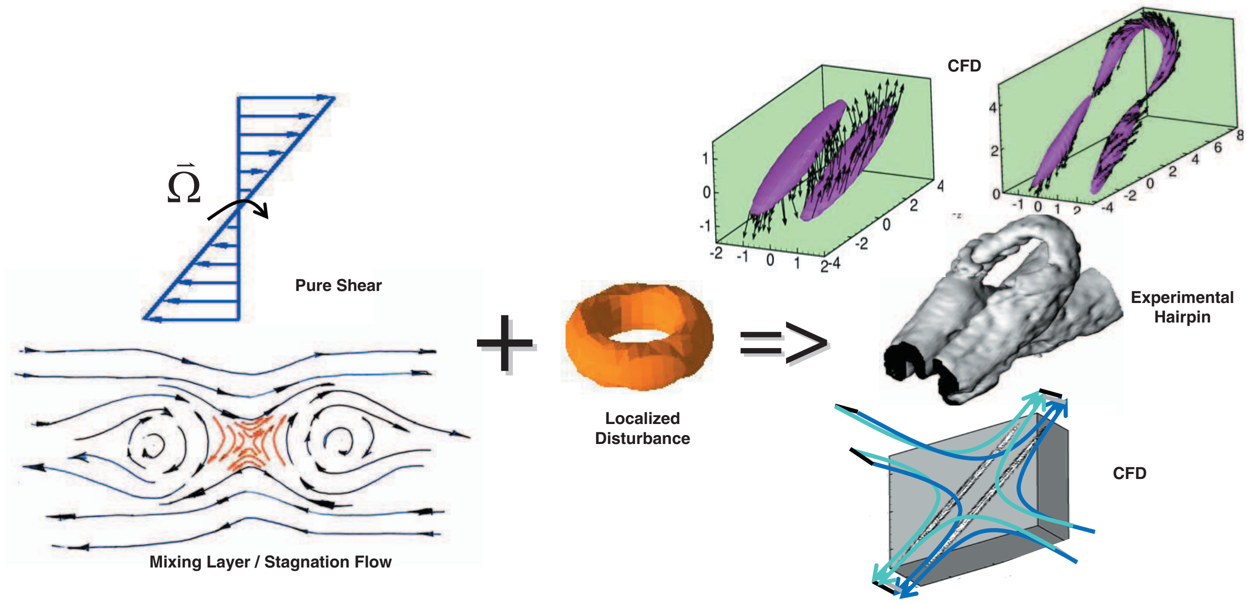

Figure 1: Effect of interaction between shear flow and a localized disturbance.

$$\frac{d\zeta_2(\boldsymbol{q},t)}{dt} = -\frac{1}{2}\,(\sigma+\Omega)\,\zeta_1(\boldsymbol{q},t) - \Omega\,\frac{k_2\left[\,k_1\,\zeta_2(\boldsymbol{q},t) - k_2\,\zeta_1(\boldsymbol{q},t)\right]}{k^2} - \eta\,k^2\,\zeta_2(\boldsymbol{q},t) + N_2(\boldsymbol{q},t),$$

and

$$\zeta_3(\boldsymbol{q},t) = -\frac{1}{q_3}\,[\,k_1\,\zeta_1(\boldsymbol{q},t) + k_2\,\zeta_2(\boldsymbol{q},t)].$$

where ζ_i is the Lagrangian vorticity in Fourier space, $\boldsymbol{k}$ the time-dependent wave vector and $\boldsymbol{q}$ its initial value. The nonlinear terms are given by the following convolution in the Lagrangian variables space:

$$N_1(\boldsymbol{q}) = \int \frac{d^3q'}{(k')^2}\left\{k_2\Big[\Big(k'_3\,\zeta_1(\boldsymbol{q}') - k'_1\,\zeta_3(\boldsymbol{q}')\Big)\,\zeta_1(\boldsymbol{q}-\boldsymbol{q}') - \Big(k'_2\,\zeta_3(\boldsymbol{q}') - k'_3\,\zeta_2(\boldsymbol{q}')\Big)\,\zeta_2(\boldsymbol{q}-\boldsymbol{q}')\Big] + \right.$$

$$\left. +k_3\left[\Big(k'_1\,\zeta_2(\boldsymbol{q}') - k'_2\,\zeta_1(\boldsymbol{q}')\Big)\,\zeta_1(\boldsymbol{q}-\boldsymbol{q}') - \Big(q'_k\,\zeta_3(\boldsymbol{q}') - k'_3\,\zeta_2(\boldsymbol{q}')\Big)\,\zeta_2(\boldsymbol{q}-\boldsymbol{q}')\right]\right\},$$

$$N_2(\boldsymbol{q}) = \int \frac{d^3q'}{(k')^2}\left\{k_1\Big[\Big(k'_2\,\zeta_3(\boldsymbol{q}') - k'_3\,\zeta_2(\boldsymbol{q}')\Big)\,\zeta_2(\boldsymbol{q}-\boldsymbol{q}') - \Big(k'_3\,\zeta_1(\boldsymbol{q}') - k'_1\,\zeta_3(\boldsymbol{q}')\Big)\,\zeta_1(\boldsymbol{q}-\boldsymbol{q}')\Big] + \right.$$

$$\left. +k_3\left[\Big(k'_1\,\zeta_2(\boldsymbol{q}') - k'_2\,\zeta_1(\boldsymbol{q}')\Big)\,\zeta_2(\boldsymbol{q}-\boldsymbol{q}') - \Big(k'_3\,\zeta_1(\boldsymbol{q}') - k'_1\,\zeta_3(\boldsymbol{q}')\Big)\,\zeta_3(\boldsymbol{q}-\boldsymbol{q}')\right]\right\}.$$

The equations are solved in the Fourier space with Lagrangian variables using a simple time stepping and fast fourier transform, with an initial condition of a localized vortical disturbance.

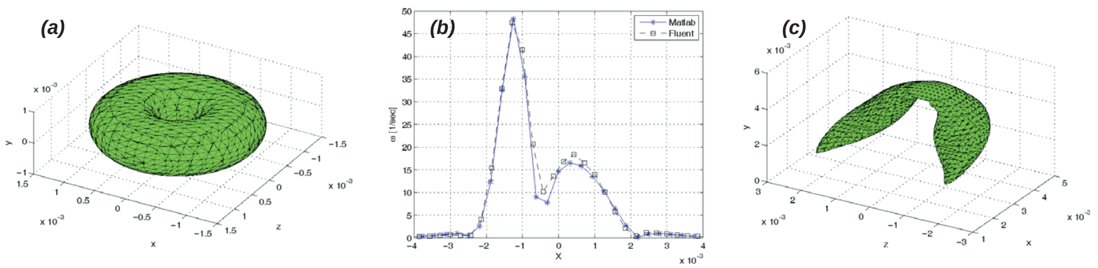

Figure 2: An initial localized Gaussian disturbance (a), and its interaction with Couette flow (b,c).

The final solution in real space is obtained by the Inverse transform given by:

$$\omega_i(\boldsymbol{r},t) = \int d^3q\, \zeta_i(\boldsymbol{q},t)\, e^{i\,[\,q_1\, x_0(\mathbf{r},\,t) + q_2\, y_0(\mathbf{r},\,t) + q_3\, z_0]},$$

where $\boldsymbol{r}_0(\boldsymbol{r},t)$ is an initial position of the fluid particle which at the instant of time t is placed at the point $\boldsymbol{r}$. The whole solution procedure is completed within minutes by using the Matlab software and a home PC.

3 Results

We choose the initial vorticity distribution in the form of a Gaussian vortex (figure 2(a)), given by:

$$\boldsymbol{\omega}(\boldsymbol{r}, t=0) = \nabla F \times \boldsymbol{\mu}, \quad F = (\pi^{1/2}\delta)^{-3} \exp\left(-r^2/\delta^2\right),$$

where r is the spherical radial co-ordinate, δ is the representative length scale of the disturbance and μ is the initial fluid impulse. The values of $\delta = 0.001[m]$ and $\eta = 10^{-6}[m^2/s]$ (corresponding to water) are taken in the calculations presented here. The strength of the initial vortex is represented by $\varepsilon = \omega_{\max}(t=0)/\Omega_*$, where $\omega_{\max}(t=0) = 0.154\mu/\delta^4$ for the Gaussian vortex and $\Omega_* = \frac{1}{2}(|\Omega| + |\sigma|)$. The magnitude of ε can be used to describe an initial vortex as linear ($\varepsilon << 1$) or nonlinear.

Figure 2(b) presents the comparison with the results when the base flow is plane *Couette* with $\Omega = \sigma = -40[1/s]$. The dashed line shows the results obtained using the commercial code Fluent by (4) and full lines by the present method for a nonlinear disturbance with $\boldsymbol{\mu} = (0, 4.9 \cdot 10^{-10}, 0)[m^4/s]$ ($\varepsilon = 1.875$). Figure 2(c) plots the iso-surfaces of vorticity magnitude ($|\boldsymbol{\omega}|/\omega_{\max} = 0.65$) for 'the Reynolds number of vortex', $Re = \Omega_*\delta^2/\eta = 40$ at $T = (t\Omega_*) = 2$, showing the evolved hairpin.

Next, an *Irrotational* (or pure extensional) base flow is considered, with $\Omega = 0$ and $\sigma = -40[1/\sec]$. Figure 3(a) shows the comparison with the published results (5) for time T=1. The match is considered quite well for this nonlinear initial vortex with $\boldsymbol{\mu} = (2.5 \cdot 10^{-10}, 0, 0)[m^4/s]$ ($\varepsilon = 1$). Figure 3(b) shows the extended structure of the vortex in real space. The plot is an iso-contour of the vorticity magnitude ($|\boldsymbol{\omega}|$). The resultant structure is reminiscent of the ribs commonly found in the braid region of a mixing layer (figure 1).

The last type of base flow represented by a homogeneous shear is the *Elliptic* flow. Figure 3(c) shows the vorticity iso-contour top view for T=4.5, superimposed on the base flow with $\Omega = 40[1/s]$, $\sigma = 32[1/s]$ and $\boldsymbol{\mu} = (4.5 \cdot 10^{-12}, 0, 0)[m^4/s]$.

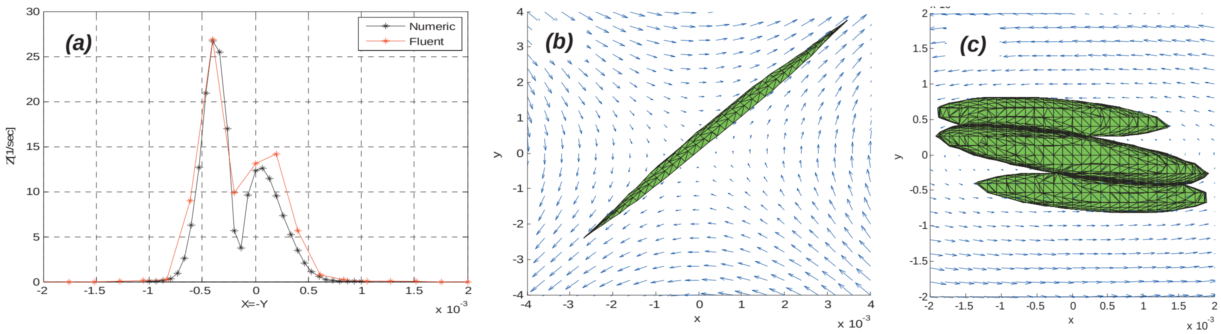

Figure 3: Interaction of a localized Gaussian disturbance with an Irrotational (a,b) and Elliptic flow (c).

4 Conclusions

The evolution of localized vortical disturbances in homogenous shear can lead to the development of key structures (e.g. CVPs and hairpin vortices) that have been observed in turbulent and transitional shear flows. We have developed an analytical-based method, which involves solution in Fourier space with Lagrangian co-ordinates, that can predict this nonlinear evolution in any linear base flows, namely Couette, Hyperbolic (in particular, pure extensional) and Elliptical flows. Comparison to available results in the literature shows high fidelity of the current method.

References

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