Proceedings of the 53rd Israel Annual Conference
on Aerospace Sciences, Tel-Aviv & Haifa, Israel,
March 6-7, 2013

Transition to turbulence in Couette flow

M. Karp and J. Cohen

*Faculty of Aerospace engineering, Technion - Israel Institute of Technology,
Haifa 32000, Israel*

This paper describes a laminar to turbulent transition scenario in Couette flow, triggered by the transient growth mechanism. An analytical derivation of a growing disturbance corresponding to counter rotating vortices embedded in Couette flow allows performing a secondary linear stability analysis of the perturbed flow. Based on the comparison with direct numerical simulation results, it is argued that a significant transient growth is not a prerequisite for transition and that the role of the initial transient growth is to generate a strong wall-normal inflectional point rather than maximal growth.

I. Introduction

Despite much progress in the field of fluid mechanics over the last century, prediction of transition to turbulence - namely, the critical Reynolds number separating the laminar and turbulent regimes, remains a puzzling issue. At first, linear stability theory (LST) was used to calculate the critical Reynolds number for given flow conditions. However, for some basic flows, such as plane Couette flow, the LST has not been successful in predicting the critical Reynolds since the flow is linearly stable (e.g. [1]). Nevertheless, experimentally transition is observed at Reynolds numbers greater than $Re \sim 360$, where Re is based on the wall velocity and half the channel height. A possible explanation for such transition phenomenon has been proposed by the inviscid mechanism of non-modal growth where the disturbance grows linearly with time [2]. This was shown for a three dimensional disturbance which does not vary in the streamwise direction. For a viscous base flow, it was shown [3] that a disturbance can achieve initial significant growth before its eventual decay due to viscous effects. This mechanism is called transient growth. Therefore, a possible explanation for transition in linearly stable flows is that a small disturbance is amplified significantly by transient growth such that nonlinear effects are triggered leading to transition. Much research has been devoted to find the initial optimal disturbances which yield the maximal growth over a given time or distance. It was shown that the maximal growth is obtained by a disturbance initially consisting of a pair of nearly streamwise independent counter rotating vortices (CVPs), e.g. [4]. Those vortices create streamwise streaks which vary along the spanwise direction (with a wavenumber β) by lifting low momentum fluid into a region of high momentum fluid and vise versa. The CVPs mainly consist of a pair of nearly parallel least stable modes, as demonstrated by [5] and [6]. Numerical simulations have shown that these streaks may undergo secondary instability [7] and can lead to transition to turbulence (e.g. [8] for pipe flow).

In this paper we show that transient growth can be achieved using only four decaying modes. Furthermore, it is argued that a significant transient growth is not a prerequisite for transition; the existence of an inflection point is at least as important. This is demonstrated by a transition scenario based on transient growth consisting of four modes and a secondary infinitesimal disturbance.

The following section describes the mathematical method, containing the analytical model, stability analysis and direct numerical simulation (DNS), followed by results and conclusions.

II. Mathematical Method

A. Analytical Model For Transient Growth

Ben-Dov *et. al.* have shown that the basic mechanism of transient growth can be explained by considering only the pair of least stable modes [5], with an analytical prediction of the time at which the maximum energy amplification of the initial disturbance is achieved. However, the maximum growth obtained by the least stable pair is not well predicted. Following [5] we propose that including another pair of modes and determining the optimal ratios between the four modes allows a good analytical prediction of the full amplification curve. The second pair should be anti-symmetric (symmetric) if the first pair is anti-symmetric (symmetric), this kind of disturbance is referred as an odd (even) disturbance, consisting of a pair of CVPs (two pairs of CVPs). Figure 1 shows a comparison between the optimal curve based on many modes (black) and an analytical curve based on only four modes (red) for an odd disturbance, $Re = 3000$ and $\beta = 1.66$. It can be seen that the analytical curve provides a good approximation for the optimal one. The initial transient growth disturbance has the following form:

$$u(y,z) = Re\left\{\sum_{m=1}^{4} A_m u_m(y) e^{i\beta z}\right\} \tag{1}$$

where u_m are the modes obtained from the LST analysis for the given spanwise wavenumber β, applied to Couette flow. The coefficient A_4 is set to be 1 and the remaining three coefficients are optimized in order to maximize the growth.

Following the discovery of transient growth, the main objective was obtaining the optimal disturbance yielding the maximal growth (e.g. [9]). However, later investigations have shown that other flow structures may lead to turbulence with lower initial thresholds (e.g. transition with oblique waves [10]). We propose that the role of the initial disturbance is mainly to generate an inflectional point in the wall-normal velocity profile. In order to demonstrate this we choose an even disturbance for which the maximum growth is about an order of magnitude lower than the corresponding odd mode (for $Re = 3000$ and $\beta = 1.66$ the maximal growth of the even disturbance is ~ 720 comparing to $\sim 10^4$ of the odd disturbance).

The importance of the inflectional point can be evaluated by performing a secondary stability analysis of the perturbed base flow (Couette flow plus CVPs). The analytical expression of the streamwise velocity and its simple dependance on the spanwise coordinate (z) allows deriving a simple form of the eigenvalue problem. The stability analysis is utilized to obtain the optimal disturbances and their corresponding growth rates.

The analytical derivations for the transient growth and the stability analysis are verified using the DNS software 'Channelflow' [11]. The DNS is also used to obtain transition scenarios using the secondary disturbances from the stability analysis.

B. Stability Analysis

A two dimensional linear stability analysis is performed to a base flow having the form $\boldsymbol{U} = (U_0(y,z), 0, 0)$ where y and z are the wall-normal and the spanwise directions, respectively (e.g. [12]). The analysis is conducted using the Floquet theory for the periodic base-flow in the spanwise direction, i.e. $U_0(y, z+2\pi\beta) = U_0(y, z)$. A necessary assumption for performing the stability analysis is that the time scale of the streak transient growth and decay is much smaller than the time scale associated with the growth of the secondary instability. This allows separation of time scales and performing the stability analysis for a particular 'frozen' time. Another assumption is that relative to the streamwise velocity, the wall-normal and spanwise velocities of the base flow are negligible.

We assume a secondary disturbance of the form:

$$\boldsymbol{q} = e^{i(\alpha x - \omega t)} \sum_{k=1}^{M_z} \hat{\boldsymbol{q}}_k(y) e^{i\beta k z} \tag{2}$$

where $\boldsymbol{q} = (u; v; w; p)$, α is the streamwise wavenumber and ω is the temporal complex eigenvalue. Substituting the disturbance in the linearized Navier-Stokes equations allows obtaining an eigenvalue problem for the calculation of the eigenvalues ω_n and the eigenfunctions $(u_n; v_n; w_n; p_n)$, where the subscript n indicates the n-th eigenvalue for given α, β and Re. The use of an analytical expression for the base flow allows convergence for relatively small values of M_z and it is found that $M_z = 5$ is sufficient in order to obtain the eigenvalues within accuracy of 0.5%. The eigenvalue problem is solved numerically using Matlab with 101 Chebyshev modes in the wall-normal direction.

C. Direct Numerical Simulation

The computation of the transition scenario is performed using Gibson's well-tested 'Channelflow' DNS software [11]. The simulation code is pseudospectral, using Fourier modes in the x and z directions and Chebyshev (collocation) modes in the y direction with no-slip and impermeability on the walls at $y = \pm 1$. The computational box contains a single wavelength in the streamwise and spanwise directions $L_x = 2\pi/\alpha, L_z = 2\pi/\beta$. The results are obtained on a grid containing $(64, 65, 64)$ points in the x, y and z directions, respectively. It is found that increasing the number of modes improves the accuracy, however, its influence on the disturbance energy is minor. In order to remove aliasing the 3/2 rule is applied so that the number of corresponding Fourier modes is $N'_{x,z} = \frac{2}{3} N_{x,z}$. The time step is chosen to obtain an initial Courant-Friedrichs-Lewy (CFL) number of 0.17.

III. Results

The disturbance growth is calculated from its energy, given by the volumetric integral:

$$G(t) = \int (u^2 + v^2 + w^2) d\boldsymbol{x}, \tag{3}$$

normalized by its initial value $G_0 = G(t = 0)$.

The transient growth of an odd disturbance obtained from the analytical derivation is simulated using the DNS for $Re = 3000$ and $\beta = 1.66$. Figure 2 shows a comparison of the DNS results (marked by the blue circles) with the linear analytical prediction based on four modes (solid yellow line) and there is a good agreement between the two. The good agreement is an outcome of the low initial amplitude ($1/Re$) used in this case, justifying the neglect of the nonlinear terms.

As the amplitude of the disturbance is increased, the nonlinear terms become more dominant and a deviation between the linear analytical prediction based on four modes (solid yellow line) and the DNS (marked by the black crosses) is observed, as evident from figure 3. The figure shows a comparison between linear theory and DNS for the higher amplitude $5/Re$. Note that both curves are normalized by their corresponding initial values, which gives the wrong impression that the nonlinear case (DNS) grows less.

The stability analysis is performed for $Re = 1000$ and a secondary disturbance having streamwise and spanwise wavenumbers of $\alpha = 1$ and $\beta = 1$, respectively. Although the maximum growth of the odd mode is ~ 800, we choose to perform the analysis for an even mode having a maximal growth of ~ 40 achieved at $t/Re \sim 0.065$ (see the red curve at figure 5). The most unstable eigenvalue for various times is shown in figure 4. At $t = 0$ the streaks have not yet been formed and the streamwise disturbance is very weak, such that the obtained eigenvalue corresponds to the eigenvalue associated with LST of pure plane Couette flow for $Re = 1000$ and $\alpha = 1$ which gives the growth rate $\omega_i = -0.1192$. As time is increased, the flow becomes less stable and instability is obtained for $t > 10$.

In order to verify the instability of the above even disturbance we have simulated a perturbed transient growth with the addition of a secondary disturbance having very small magnitude. The secondary disturbance is obtained by the stability analysis for the time $t = 20$. This resulted in transition to turbulence. The resulting energy growth is shown in figure 5 by the blue line. The linear transient growth is shown as reference by the red line. It can be seen that the disturbance energy follows the transient growth up to $t/Re \approx 0.04$ when the growth becomes significant and the flow undergoes transition.

The importance of the wall-normal inflection point, circled in the left of figure 6, can be understood by examining the shape of the streamwise velocity eigenfunction obtained using the secondary stability analysis. As shown in the right of figure 6, the secondary disturbance is concentrated mainly around $(y, z) = (0, \pi/\beta)$ where the base flow possesses a wall-normal inflectional point. This suggests that the role of the initial transient growth is to generate a strong wall-normal inflectional point rather than maximal growth.

IV. Conclusions

It is shown that four decaying modes are sufficient to approximately follow the transient growth scenario in plane Couette flow. This enables us to follow the stability of the evolving flow during transient growth. The results suggest that the role of the initial transient growth is to generate a strong wall-normal inflectional point rather than maximal growth.

Acknowledgments

This research was supported by the Israeli Science Foundation under Grant No. 1394/11.

References

[1] Schmid, P. J. and Henningson, D. S., *"Stability and transition in shear flows"*, Springer-Verlag New York, 2001.

[2] Ellingsen, T. and Palm, E., "Stability of linear flow," *Phys. Fluids*, Vol. 18, 1975,

pp. 487–488.

[3] Hultgren, L. S. and Gustavsson, L. H., “Algebraic growth of disturbances in a laminar boundary layer,” *Phys. Fluids*, Vol. 24, 1981, pp. 1000.

[4] Gustavsson, L. H., “Energy growth of three-dimensional disturbances in plane Poiseuille flow,” *J. Fluid Mech.*, Vol. 224, 1991, pp. 241.

[5] Ben-Dov, G., Levinski, V., and Cohen, J., “On the mechanism of optimal disturbances: The role of a pair of nearly parallel modes,” *Phys. Fluids*, Vol. 15, 2003, pp. 1961–1972.

[6] Cohen, J., Philip, J., and Ben-Dov, G., “Aspects of linear and nonlinear instabilities leading to transition in pipe and channel flows,” *Phil. Trans. R. Soc. A*, Vol. 367, 2009, pp. 509–527.

[7] Zikanov, O. Y., “On the stability of pipe Poiseuille flow,” *Phys. Fluids*, Vol. 8, 1996, pp. 2923–2932.

[8] Meseguer, A., “Streak breakdown instability in pipe Poiseuille flow,” *Phys. Fluids*, Vol. 15, 2003, pp. 1203–1213.

[9] Reddy, S. C. and Henningson, D. S., “Energy growth in viscous channel flow,” *J. Fluid Mech.*, Vol. 252, 1993, pp. 209–238.

[10] Lundbladh, A., Henningson, D., and Reddy, S., “Threshold amplitudes for transition in channel flows,” *Transition, turbulence and combustion*, Vol. 1, 1994, pp. 309–318.

[11] Gibson, J. F., “Channelflow: A spectral Navier-Stokes simulator in C++,” Tech. rep., U. New Hampshire, 2012, `Channelflow.org`.

[12] Reddy, S. C., Schmid, P. J., Baggett, J. S., and Henningson, D. S., “On the stability of streamwise streaks and transition thresholds in plane channel flows,” *J. Fluid Mech.*, Vol. 365, 1998, pp. 269.

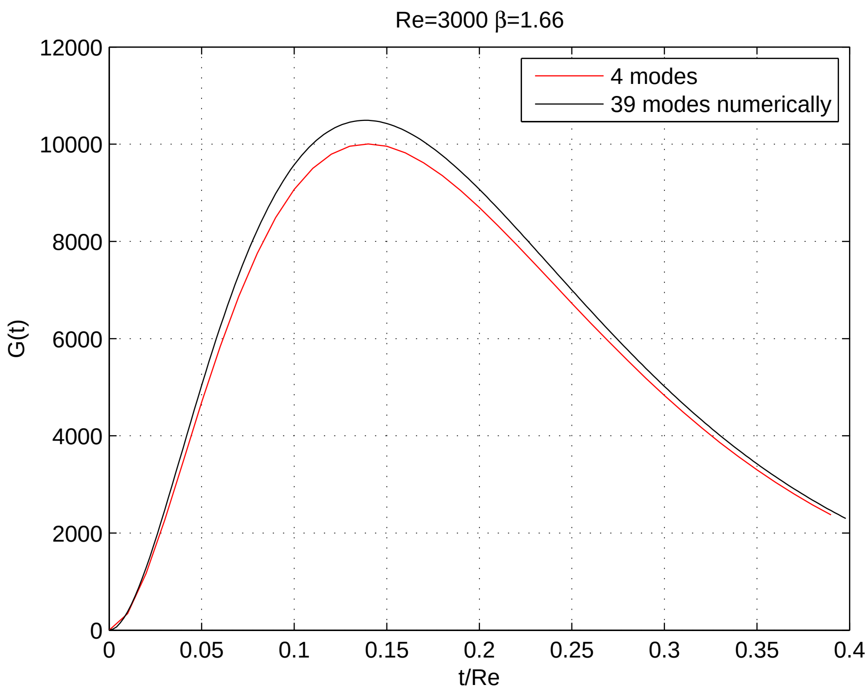

Figure 1. Transient growth for an odd disturbance, $Re = 3000$ and $\beta = 1.66$. Comparison between the optimal disturbance and an analytical prediction based on four modes

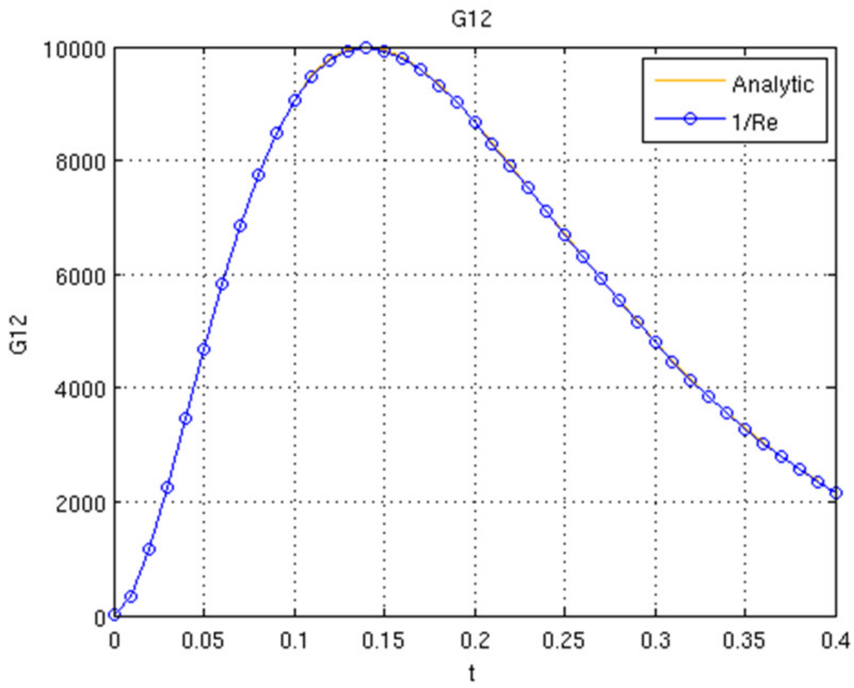

Figure 2. Transient growth for an odd disturbance, $Re = 3000$ and $\beta = 1.66$. Comparison between linear analytical prediction based on four modes (solid yellow line) and DNS (marked by the blue circles) for the disturbance having the amplitude $1/Re$

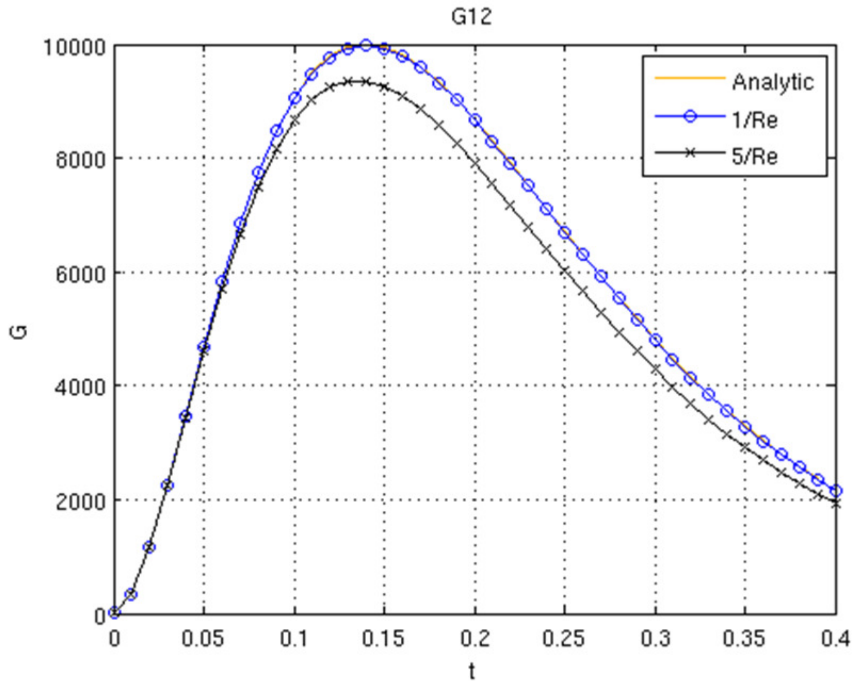

Figure 3. The energy gain for an odd disturbance, $Re = 3000$ and $\beta = 1.66$. Comparison between linear analytical prediction based on four modes (solid yellow line) and DNS (marked by the black crosses) for the disturbance having the amplitude $5/Re$

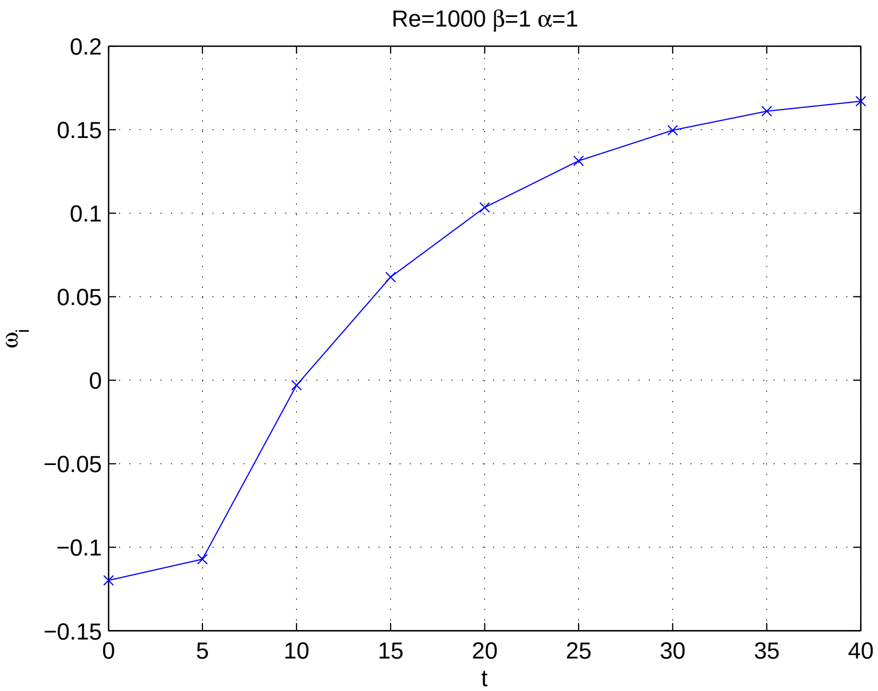

Figure 4. Growth rate of the least stable mode at various times, obtained by secondary stability analysis for an even disturbance with $Re = 1000, \alpha = 1, \beta = 1$

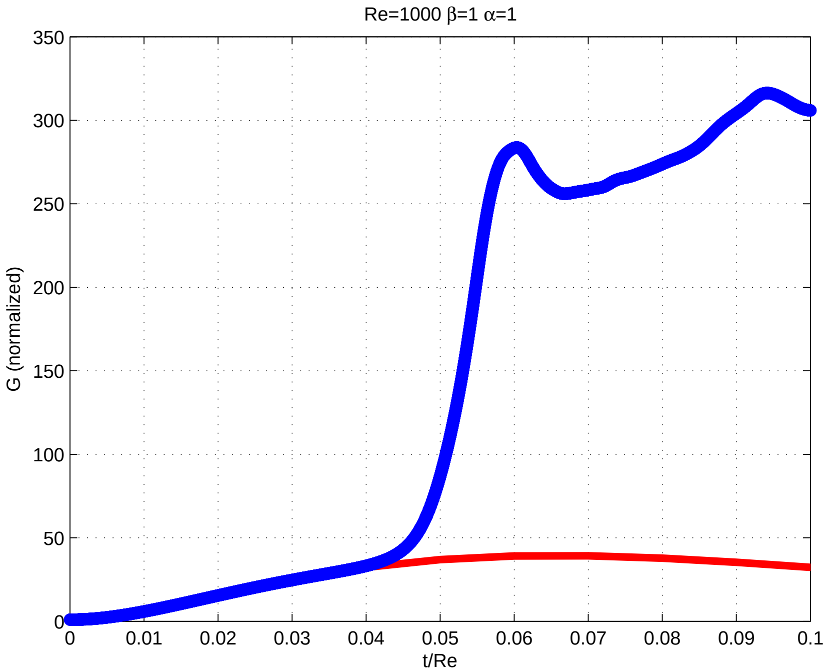

Figure 5. A transition scenario for an even disturbance $Re = 1000, \alpha = 1, \beta = 1$. The unperturbed transient growth is given for reference

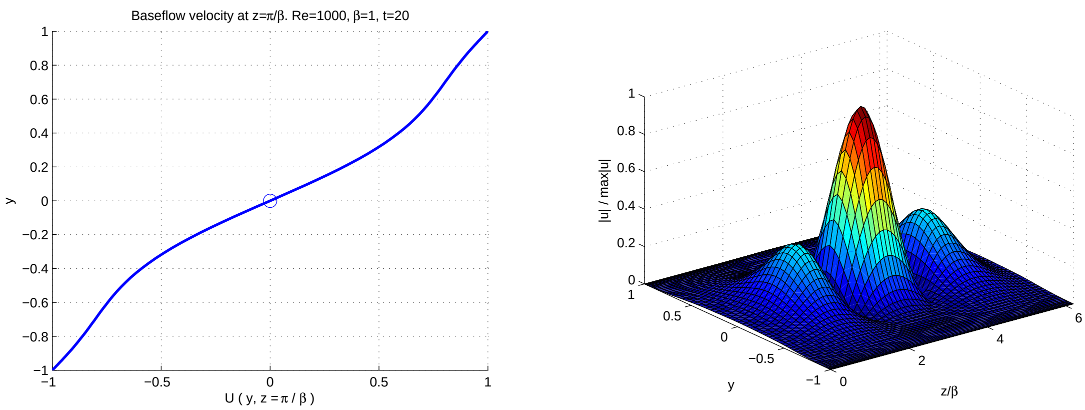

Figure 6. Left: base flow velocity at $z = \pi/\beta$ for $Re = 1000, \beta = 1$ and $t = 20$, the wall-normal inflectional point is circled. Right: streamwise secondary disturbance at $t = 20$ for $Re = 1000, \alpha = 1$ and $\beta = 1$